Implementability of Liouville evolution, Koopman and Banach-Lamperti theorems in classical and quantum dynamics

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Abstract

We extend the concept of implementability of semigroups of evolution operators associated with dynamical systems to quantum case. We show that such an extension can be properly formulated in terms of Jordan morphisms and isometries on non-commutative L^p spaces. We focus our attention on a non-commutative analog of the Banach-Lamperti theorem.

1 Introduction

Operator theory and the associated theory of semigroups proved to be one of the most successful methods elaborated for the study of dynamical systems. The idea of using operator theory is due to Koopman who replaced the time evolution S_t of single points

from a phase space Ω by the time evolution of the corresponding Koopman operators V_t defined as

$$V_t f(\omega) = f(S_t\omega)\,, \qquad f \in L^2(\Omega),\ \omega \in \Omega\,.$$

Koopman [1] introduced these operators in 1931 in order to study the ergodic properties of dynamical systems using the powerful tools of operator theory. This approach has been extensively used thereafter in statistical mechanics and ergodic theory. The objects under consideration are Koopman operators regarded as operators on L^p spaces, $1 \leq p \leq \infty$, and their adjoints called Frobenius-Perron operators. Frobenius-Perron operators describe, in particular, the evolution of probability densities defined on the phase space Ω.

The application of operator theory to dynamical systems simplifies the study of their ergodic properties such as ergodicity, mixing and exactness, as well as Kolmogorov systems, which is the basis of the modern theory of chaos [2, 3]. Particularly important is the spectral analysis of evolution operators that enables to extract important information about their dynamical properties such as for example, the rate of the convergence to equilibrium. Recent results obtained by the Brussels group (see [4] and the references therein) show that for unstable dynamical systems there exist spectral decompositions of the evolution operators in terms of resonances and resonance states, which appear as eigenvalues and eigenprojections of the evolution operators. Another powerful method for the study of unstable dynamical systems is based on the concept of a time operator [5, 6], which is defined as a selfadjoint operator T associated with the evolution semigroup V_t through the commutation relation

$$TV_t = V_t T + tV_t\,.$$

The dynamical systems that admit time operators are highly unstable like Kolmogorov or exact systems. Nevertheless, the knowledge of the eigenvectors of T amounts to a probabilistic solution of the prediction problem for the dynamical system described by the semigroup $\{V_t\}$.

The time operator method serves to elucidate the problem of irreversibility in statistical physics which is related to the understanding of the relation between reversible dynamical laws and the observed entropy increasing evolutions. Misra, Prigogine and Courbage [7] showed that the unitary evolution U_t of a Kolmogorov system can be intertwined with a Markov semigroup W_t, $t \geq 0$, through a non-unitary transformation Λ:

$$W_t\Lambda = \Lambda U_t\,, \quad t \geq 0\,. \tag{1}$$

The intertwining transformation Λ in the Misra-Prigogine-Courbage approach is a non-increasing function of the time operator.

The evolution operators that arise from point transformations of a phase space are often modified, like in the Misra-Prigogine-Courbage theory of irreversibility, leading to new evolution semigroups that need not to be related with the underlying point dynamics. The natural question is: Are such operators associated with other point transformations? In other words, we ask if, for example, modifications made on the level of evolution operators correspond to some modifications on the level of trajectories in the phase space. In a more general setting, we ask which linear operators on L^p spaces are implementable by point transformations? A related question is: Which time evolutions of states of physical systems that are described in terms of a semigroup $\{W_t\}$ of maps on an L^p-space can be induced by Hamiltonian flows? There are some partial answers to the above questions that will be presented below.

In quantum mechanics we can also distinguish two levels of evolution of states and observables that can be expressed in terms of evolution operators and semigroups. Thus similar questions, as in the classical case, concerning implementability can be raised. The quantum case is however more complex both technically and conceptually. The evolution operators act on non-commutative L^p spaces that have much more complex structure and require sophisticated tools from operator algebras theory. Secondly the very basic concept of implementability requires clarification.

In this article we formulate the implementability in quantum case and prove analogs of some classical results. The paper is organized as follows. In Section 2 we give an overview of the results on implementability in classical case. Section 3 contains an introduction to non-commutative L^p spaces and quantum dynamics. The formulation of quantum implementability and our main results are in Section 4.

2 Classical case

Let (Ω, Σ, μ) be a measure space with a finite measure μ. A one parameter evolution semigroup $\{S_t\}$ of measurable transformations of the space Ω defines a dynamical system. The variable t signifies time and is continuous for flows and discrete for cascades. For reversible systems $\{S_t\}$ is a group of automorphisms of Ω. The space Ω equipped with a measure structure is called the phase space and measure μ represents in the case $\mu(\Omega) = 1$ an equilibrium distribution.

The phase functions f evolve according to the Koopman operators

$$V_t f(\omega) = f(S_t\omega) \ , \quad \omega \in \Omega \, . \tag{2}$$

The Koopman operators are isometries on the Banach space $L^p = L^p(\Omega, \Sigma, \mu)$, $p \geq 1$, of p-integrable functions provided that $\{S_t\}$ are measure preserving transformations. If S_t are automorphisms the Koopman operators restricted to the Hilbert space L^2 are unitary.

Consider the case of discrete time $t = 1, 2, \ldots$ when the evolution semigroup $\{V_t\}$ is determined by a single transformation S

$$V_n = V^n \quad \text{and} \quad Vf(\omega) = f(S\omega) \, .$$

The relation of the point dynamics with the Koopman operators is clarified by asking the question: What types of isometries on L^p spaces are implementable by point transformations? For L^p spaces with $p \neq 2$, all isometries induce underlying point transformations. Such theorems on the implementability of isometries on L^p spaces, $p \neq 2$, are known as Banach-Lamperti theorems [8, 9]. The converse to Koopman's lemma in the case $p = 2$, which holds under the additional assumption that the isometry on L^2 is positivity preserving, can be found in [10]. The result is that an isometry V is implementable by a necessarily measure preserving transformation S

$$Vf(\omega) = f(S\omega) \ , \quad \omega \in \Omega \, .$$

The just quoted results on the relations between the point dynamics and Koopman's maps on L^p-spaces gain additional interest if we realize that L^p-structures can be used in the analysis of a very large class of dynamical systems. To describe briefly how L^p-techniques may be used for such analysis let us assume that the state space Ω is a compact metric space and Σ is the σ-algebra of Borel sets. Let us consider a (homogeneous) Markov process on Ω. Then, the Kopmman's-like construction leads to the well defined semigroup W_t on the set $L^\infty(\Omega)$ of all bounded measurable functions on Ω. Let us restrict ourselves to Markov-Feller processes, i.e. such processes that $W_t : C(\Omega) \to C(\Omega)$, where $C(\Omega)$ denotes the set of all continuous complex-valued functions on Ω. The role of the set $C(\Omega)$ for the quantization procedure will be explained in the next section. Then, one can show that any Markov-Feller process induces a positivity preserving semigroup on $C(\Omega)$ and, conversely, with each such a semigroup is associated a Markov-Feller process. Moreover, such semigroups on $C(\Omega)$

can be extended to a semigroup of contractions on $L^p(\Omega, \mu)$, where $p \geq 1$ and a measure μ is time invariant (here, μ is called a time invariant measure if $\mu(W_t f) = \mu(f)$, $f \in C(\Omega)$). We recall that Markov-Feller processes constitute an important tool in the description of real systems of interacting particles (see [11]). Consequently, we can study stochastic processes in real physical models in terms of semigroups on L^p-spaces. The important point to note here is that such an approach yields the possibility of studying, in an effective way, various ergodic properties of the considered processes, e.g. the question of convergence to equilibrium, question of spectral gaps, hypercontractivity (i.e., a set of ideas in field theory, important for determination of the best constants in classical inequalities and bounds on semigroup kernels, cf. [12]), and finally to utilize various types of inequalities (e.g. log Sobolev and Nash inequalities, see [13] as well as [12] and references therein).

Before addressing the question of implementability of Misra-Prigogine-Courbage semigroups introduced in the previous section let us first recall some basic facts. Consider an abstract dynamical flow given by the quadruple $(\Omega, \Sigma, \mu, \{S_t\})$, where $\{S_t\}$ is a group of one-to-one μ invariant transformations of Ω and either $t \in \mathbf{Z}$ or $t \in \mathbf{R}$. The invariance of the measure μ implies that the transformations U_t

$$U_t\, \rho(\omega) = \rho(S_{-t}\omega), \quad \rho \in L^2$$

are unitary operators on L^2. Generally, U_t is an isometry on the space L^p, $1 \leq p \leq \infty$. Let us point out the following, very important properties of U_t as operators on L^1:

(a) $U_t \rho \geq 0$ if $\rho \geq 0$,

(b) $\int_\Omega U_t \rho \, d\mu = \int_\Omega \rho \, d\mu$, for $\rho \geq 0$,

(c) $U_t 1 = 1$.

An abstract operator W on L^1 which satisfies conditions (a)–(c) is called *doubly stochastic operator*.

The Misra-Prigogine-Courbage theory of irreversibility [7] (see [14] for its generalized version) proposes to relate the group $\{U_t\}$, considered on the space L^1, to the irreversible semigroup W_t, $t \geq 0$, through a nonunitary, doubly stochastic operator Λ:

$$W_t \Lambda = \Lambda U_t \, , \quad t \geq 0 \, ,$$

The operators W_t, which also form a doubly stochastic semigroup on L^1, should tend strongly to the equilibrium state, as $t \to \infty$, on some subset of admissible densities.

A dynamical system for which such a construction is possible is called *intrinsically random* and the conversion of the reversible group $\{U_t\}$ into the irreversible semigroup $\{W_t\}$ through a nonunitary transformation Λ is called a *change of representation.*

So far all known constructions of the operator Λ have been done for dynamical systems which are K-flows. Let us recall that a dynamical system is a K-flow if there exists a sub-σ-algebra Σ_0 of Σ such that for $\Sigma_t = S_t(\Sigma_0)$ we have

(i) $\Sigma_s \subset \Sigma_t$, for $s < t$

(ii) $\sigma(\cup_{t\in\mathbf{R}}\Sigma_t) = \Sigma$

(iii) $\cap_{t\in\mathbf{R}}\Sigma_t = \Sigma_{-\infty}$ – the trivial σ-algebra, i.e. the algebra of sets of measure 0 or 1

where $\sigma(\cup_{t\in\mathbf{R}}\Sigma_t)$ stands for σ-algebra generated by Σ_t, $t \in \mathbf{R}$. The main idea of the construction of Λ is the following. With any K-flow we can associate a family of conditional expectations $\{E_t\}$ with respect to the σ-algebras $\{\Sigma_t\}$ (projectors if we confine ourselves to the Hilbert space L^2). These projectors determine the time operator T:

$$T = \int_{-\infty}^{+\infty} t dE_t \ . \tag{3}$$

Then Λ is defined, up to constants, as a function of the operator T:

$$\Lambda = f(T) + E_{-\infty} \, , \tag{4}$$

where $E_{-\infty}$ is the expectation (projection on constants). The function f is assumed to be positive, non increasing, $f(-\infty) = 1$, $f(+\infty) = 0$ and such that lnf is concave on $\mathbf{R}$.

The Markov operators W_t are of the form

$$W_t = \left(\int_{-\infty}^{\infty} \frac{f(s)}{f(s-t)} \, dE_s + E_{-\infty} \right) U_t \, . \tag{5}$$

It should be clear that each operator U_t is the Frobenius-Perron operator associated with S_t and thus it is the adjoint of the corresponding Koopman operator. The operators W_t preserve the property of double stochasticity characteristic to Frobenius-Perron operators. Therefore the question is: are W_t Frobenius-Perron operators associated with some measure preserving transformations $\tilde{S}_t$ or, equivalently, is the adjoint W_t^* the Koopman operator

$$W_t^* f(\omega) = f(\tilde{S}_t \omega) \, .$$

As we have shown in Ref. [15] the answer to this question is in general negative. Only the choice of Λ as a coarse graining projection gives implementability [16].

3 Quantum case - non-commutative L^p-spaces

Passing to quantum theory it is convenient to rewrite the previously described scheme in terms of quantum "phase" space. The underlying philosophy is based on the general observation that various categories of spaces, in particular the classical phase space Ω, can be completely described by the (commutative) algebras of functions on them (the phase space Ω by the algebra of continuous functions $C(\Omega)$). The idea of (algebraic) quantization then is that the corresponding non-commutative algebra (C^*-algebra $\mathcal{A}$) may be viewed as an algebra of functions on a virtual "non-commutative space" ("quantum phase space" in our case). Such approach has proven to be very powerful in contemporary mathematics: for instance the analysis of the algebra of all continuous functions on a topological group led to the notion of quantum group. Moreover, this approach is a starting point for studying "geometrical properties" of non-commutative algebras (cf. [17, 18]).

Within that scheme, we are able to discuss the relation between point dynamics and Koopman's operators for the Quantum Mechanics setting. Namely, according to the above strategy point dynamics may be viewed as a one parameter family of maps $\hat{S}_t$ on a C^*-algebra $\mathcal{A}$. Clearly, in this way we also include general quantum Markov-Feller dynamics into the considered framework for quantizing dynamical systems. Further, the Koopman's maps $\hat{V}_t$ will be defined on non-commutative $L^p(\mathcal{A})$ spaces which are quantum analogues of classical $L^p(\Omega, \Sigma, \mu)$ spaces. The relationship between $\hat{S}_t$ and $\hat{V}_t$ expresses the implementability of Liouville evolution for quantum systems.

To implement the just given programme we start with the algebraic reformulation of the theory of quantum dynamical systems. To this end we note that observables in quantum mechanics are described by selfadjoint operators on some Hilbert space $\mathcal{H}$ and physical states by positive tracial operators on $\mathcal{H}$. In the mathematical formalism it is more convenient to consider a von Neumann algebra $\mathcal{M}$ as the algebra of observables and its dual $\mathcal{M}^*$ as the algebra of states. Thus, in the algebraic reformulation of the classical dynamical system $(\Omega, \Sigma, \mu, \{S_t\})$ we consider the commutative W^*-algebra $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$. Then, the semigroup of Koopman operators will be replaced by the semigroup of homomorphisms of $\mathcal{A}$ which are given by the formula

$$[\alpha_t(a)](f)(\omega) = a(S_t\omega)f(\omega)\,,$$

for each function $a \in \mathcal{A} \subset \mathcal{B}(L^2)$ (i.e. a is treated as a bounded operator on L^2) and $f \in L^2$.

Turning to the quantum case, let us consider as a non-commutative analog of a probability space (Ω, Σ, μ) the triple $(B(\mathcal{H}), \mathcal{H}, \varrho)$ where $\mathcal{H}$ is a separable Hilbert space, $B(\mathcal{H})$ is the set of all linear bounded operators on $\mathcal{H}$ and ϱ is a density matrix. Let us assume that ϱ is an invertible operator, which implies that $\omega(\cdot) = Tr\{\varrho \,\cdot\}$ is a faithful state on $B(\mathcal{H})$. In physical terms, ϱ can represent, for example, a Gibbs state at a temperature β. Suppose that the dynamics of the system is given in the Heisenberg picture, i.e. the time evolution of the system is given by a one-parameter family of maps $\alpha_t : B(\mathcal{H}) \to B(\mathcal{H})$. More precisely, the equivalence of the Schroedinger and Heisenberg picture for reversible dynamics says, [19], that the dynamics of observables can be given by a one parameter group α_t of Jordan automorphisms (that is linear, $*$-preserving, one-to-one and onto maps defined on a C^*-algebra $\mathcal{C}$ such that $\alpha_t(A^2) = \alpha_t(A)^2$ for $A \in \mathcal{C}$).

Treating $(\mathcal{B}(\mathcal{H}), \mathcal{H}, \varrho, \alpha_t)$ as a quantum analogue of a classical quadruple $(\Omega, \Sigma, \mu, S_t)$ we will introduce basic examples of quantum L^p-spaces as follows. Observe first that the set of all Hilbert-Schmidt operators $\mathcal{F}_{H-S}$, on the space $\mathcal{H}$, has a Hilbert space structure with the inner product given by

$$(a, b) = Tr\{a^* b\}\,, \quad a, b \in \mathcal{F}_{H-S}\,.$$

Therefore $\mathcal{F}_{H-S}$ can be considered as the quantum L^2-space associated with "the quantum uniform measure":

$$\omega_0(\cdot) \equiv Tr\{\mathbf{1}\cdot\}\,. \tag{6}$$

Analogously, the set of all tracial operators (density matrices) $\mathcal{F}_T$ can be regarded as the quantum L^1-space associated with the measure (6). The corresponding norms for these spaces are:

$$||\cdot||_p \equiv (Tr|\cdot|^p)^{\frac{1}{p}}\,, \quad p = 1, 2\,.$$

Let us generalize the idea of quantum L^p spaces to an arbitrary "quantum measure" $\omega(\cdot) \equiv Tr\{\varrho\,\cdot\}$ on $\mathcal{B}(\mathcal{H})$. Let us fix $A_0 \geq 0$ and put

$$A_\varrho \equiv \varrho^{\frac{1}{2p}} A_0 \varrho^{\frac{1}{2p}}\,,$$

where $p = 1, 2$. Observe that

$$A_\varrho \leq ||A_0|| \varrho^{\frac{1}{p}}$$

and

$$Tr A_\varrho^p = Tr A_\varrho^{\frac{p-1}{2}} A_\varrho A_\varrho^{\frac{p-1}{2}} \leq ||A_0|| Tr\{A_\varrho^{\frac{p-1}{2}} \varrho^{\frac{1}{p}} A_\varrho^{\frac{p-1}{2}}\} \leq ||A_0||^p Tr\varrho \leq \infty\,.$$

The notation was chosen in such a way that it is easy to generalize the result to other $p's$. Let us note that for an arbitrary operator $A \in \mathcal{B}(\mathcal{H})$

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i} \equiv A_h + iA_a \equiv A_h^+ - A_h^- + iA_a^+ - iA_a^- ,$$

where $A_h^+, A_h^-, A_a^+, A_a^-$ are positive operators. Therefore

$$\begin{aligned}(Tr|\varrho^{\frac{1}{2p}} A \varrho^{\frac{1}{2p}}|^p)^{\frac{1}{p}} &\leq (Tr|\varrho^{\frac{1}{2p}} A_h^+ \varrho^{\frac{1}{2p}}|^p)^{\frac{1}{p}} + (Tr|\varrho^{\frac{1}{2p}} A_h^- \varrho^{\frac{1}{2p}}|^p)^{\frac{1}{p}} \\ &+ (Tr|\varrho^{\frac{1}{2p}} A_a^+ \varrho^{\frac{1}{2p}}|^p)^{\frac{1}{p}} + (Tr|\varrho^{\frac{1}{2p}} A_a^- \varrho^{\frac{1}{2p}}|^p)^{\frac{1}{p}} \\ &< \infty ,\end{aligned}$$

which implies that we can relate to each $A \in \mathcal{B}(\mathcal{H})$ a trace class operator:

$$\mathcal{B}(\mathcal{H}) \ni A \mapsto |\varrho^{\frac{1}{2p}} A \varrho^{\frac{1}{2p}}|^p \in \mathcal{F}_T , \tag{7}$$

for $p \in \{1, 2\}$, and for any fixed (arbitrary) density matrix ϱ. Consequently, it is easy to see that

$$||A||_p \equiv (Tr|\varrho^{\frac{1}{2p}} A \varrho^{\frac{1}{2p}}|^p)^{\frac{1}{p}} \tag{8}$$

is a well defined norm on $\mathcal{B}(\mathcal{H})$. The Banach space defined as the completion of $\mathcal{B}(\mathcal{H})$ in the norm (8) will be denoted as $L^p(\mathcal{B}(\mathcal{H}), \omega)$. As expected, the $L^2(\mathcal{B}(\mathcal{H}), \omega)$ is a Hilbert space with the scalar product

$$< A, B > \equiv Tr(\varrho^{\frac{1}{2}} A^* \varrho^{\frac{1}{2}} B) \tag{9}$$

The above result saying that (8) is a well defined norm on $\mathcal{B}(\mathcal{H})$ can be easily extended to an arbitrary $p \geq 1$ (see [20, 21], and [22] for details). Moreover, it can be shown that the spaces $L^p(\mathcal{B}(\mathcal{H}), \omega)$ and $L^q(\mathcal{B}(\mathcal{H}), \omega)$, with $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, are duals of each other. One can also introduce the space $L^\infty(\mathcal{B}(\mathcal{H}), \omega)$ as the dual to $L^1(\mathcal{B}(\mathcal{H}), \omega)$. Such a construction ensures that we have

$$L_p(\mathcal{B}(\mathcal{H}), \omega) \subseteq L_q(\mathcal{B}(\mathcal{H}), \omega) \tag{10}$$

for $1 \leq p \leq q \leq \infty$. Finally, note that the one parameter family of Banach spaces $\{L^p(\mathcal{B}(\mathcal{H}), \omega)\}_{p \geq 1}$ forms a so-called interpolating scale, i.e. the Banach spaces $L^p(\mathcal{B}(\mathcal{H}), \omega)$ with $1 \leq p \leq \infty$ are interpolating spaces between $L^1(\mathcal{B}(\mathcal{H}), \omega)$ and $L^\infty(\mathcal{B}(\mathcal{H}), \omega)$. In particular, interpolation theory (like in the classical case) is also available in this case. Therefore, a large number of "classical" L^p-estimates is also available for quantum dynamical systems (cf. [23]).

As an example, let us reconsider a very special case in the above construction. Namely, instead of the state ω let us take $\omega_0 \equiv Tr\{\mathbf{1}\cdot\}$. In mathematical terms we replace state by the weight $Tr(\cdot)$. Then, the repetition of the above argument leads to

$$L^p(\mathcal{B}(\mathcal{H}), Tr) \equiv \{A \in \mathcal{B}(\mathcal{H}); ||A||_p = (Tr|A|^p)^{\frac{1}{p}} < \infty\}. \tag{11}$$

It is easy to recognize that we get all p-Schatten classes. In other words, the trace class operators as well as the Hilbert-Schmidt operators, mentioned at the beginning of this section, constitute special cases of quantum L^p-spaces. We recall that these spaces have been used for the study of various problems of quantum statistical physics and that such an approach is called the quantum Liouville space technique (see [24]). For a slightly different definition of L^2-spaces associated with a quantum state and their applications to "probabilistic" descriptions of quantum systems see Chapter II in [25].

Before proceeding with the construction of non-commutative Koopman's operators let us summarize here various general L^p-spaces that we will need in the next section. We start with the observation that a general formulation of a quantum schema deals with a general W^* (or even C^*) algebra $\mathcal{A}$ describing individual properties of a fixed physical system (see [26, 27, 28]). In particular, the basic procedure of statistical mechanics, the thermodynamic limit, leads to a C^*-algebra $\mathcal{A}$ which can be very different from that of $\mathcal{B}(\mathcal{H})$. Consequently, general quantum L^p spaces corresponding to a quantum system should be based on a general C^*-algebra. However, such a general construction of non-commutative L^p-spaces is rather involved (see [22, 23, 27, 29, 30, 31]). Nevertheless, such a general scheme has proved to be very useful for the description of concrete models with quantum Markov dynamics (see [22, 30, 31, 32, 33, 34]).

As it is not our purpose to study here the mathematical questions related to a construction of general L^p-spaces we shall restrict ourselves to the case of a von Neumann algebra $\mathbf{M}$ with a faithful semifinite normal (*fsn*) trace φ. Clearly, $\mathcal{B}(\mathcal{H})$ has this property, i.e the case of Dirac's quantum mechanics will be included. Therefore, we can consider the pair $\{\mathbf{M}, \varphi\}$ consisting of a von Neumann algebra and *fsn* trace. Let ω be a normal linear functional on $\mathbf{M}$. Then (see [35]) ω is of the form $\omega(a) = \varphi(Ra)$, $a \in \mathbf{M}$, where R is an L^1-integrable (so $\varphi(|R|) < \infty$) uniquely determined non-singular positive operator. Define

$$||A||_p \equiv (\varphi|R^{\frac{1}{2p}} A R^{\frac{1}{2p}}|^p)^{\frac{1}{p}} \tag{12}$$

It can be proved that $||\cdot||_p$, $p \geq 1$, is a norm on $\mathbf{M}$ (see [20, 21]). The completion of $\mathbf{M}$ with respect to this norm again leads to the non-commutative $L^p(\mathbf{M}, \omega)$ Banach space (see [20, 21, 35, 36]) with all the listed properties of $L^p(\mathcal{B}(\mathcal{H}), \omega)$-spaces. Consequently, we got non-commutative L^p-spaces which can be associated with a large family of quantum dynamical systems.

4 The converse of Quantum Banach-Lamperti theorem

We have seen that the Koopman's construction (2) gives a well defined *bounded* map V_t on L^p-spaces, so V_t can be considered as an "integrable" map. Our first observation is that we have analogous situation in the non-commutative framework. Namely, let us assume that $T : \mathbf{M} \to \mathbf{M}$ is a linear bounded map. Here and subsequently, $(\mathbf{M}, \omega)$ denotes a semifinite von Neumann algebra and a state on it respectively. Denote by ι_p the imbedding of $\mathbf{M}$ into $L^p(\mathbf{M}, \omega)$ and define operator $T^{(p)} : L^p \to L^p$ by formula:

$$T^{(p)}(\iota_p(a)) = \iota_p(Ta) \quad a \in \mathbf{M} \tag{13}$$

We say that T is p-integrable (with respect to ω) if the induced operator $T^{(p)}$ is L^p-bounded, in which case we denote its unique extension to $L^p(\mathbf{M}, \omega)$ by the same letter. We have the following useful, and in fact very general, criterion for quantum integrability (see [37]). Let $T : \mathbf{M} \to \mathbf{M}$ be a normal, positivity preserving linear map. Then T is integrable with respect to ω if and only if $T_*\omega \leq \mathrm{Const} \circ \omega$ where T_* denotes the predual map. We want to add that we dropped here the "pth" in the word integrability as one can show that p-integrability implies r-integrability for $r \geq p$ (see [37]). In particular, if $J : \mathbf{M} \to \mathbf{M}$ is a Jordan automorphism satisfying $\omega \circ J \equiv \omega$ then $J^{(p)}$ is an integrable map. One can even show that $J^{(p)}$ is an isometry.

Having clarified this point let us turn to a quantum analogue of Banach-Lamperti result (another approach to that question was recently given in Ref. [38]). Let $\{\mathbf{M}, \varphi\}$ be a von Neumann algebra with *fsn* trace and let $L^p(\mathbf{M}, \varphi)$, $p \geq 1$, be the corresponding quantum L^p-space. Assume that T, $T : L^p(\mathbf{M}, \varphi) \to L^p(\mathbf{M}, \varphi)$ is a linear map. Then, (see [39], and [40] for the most general case), T is L^p-isometry of $L^p(\mathbf{M}, \varphi)$ *onto* itself if and only if

$$T(x) = WBJ(x), \quad x \in L^p(\mathbf{M}, \varphi) \cap \mathbf{M} \tag{14}$$

where $W \in \mathbf{M}$ is unitary, B a selfadjoint operator affiliated with the center of $\mathbf{M}$ and

J a normal Jordan isomorphism mapping $\mathbf{M}$ onto itself, such that

$$\varphi(X) = \varphi(B^p J(X)), \quad \text{for} \quad \text{all} \quad X \in \mathbf{M}, \quad X \geq 0. \tag{15}$$

Formula (14) is nothing but the statement that any L^p-isometry T of $L^p(\mathbf{M}, \varphi)$ onto itself is implemented by a Jordan morphism J which is multiplied by operators W and B.

In physics, especially in statistical physics, we are usually interested in L^p-spaces associated with a finite measure. In other words we want to associate the L^p-space with the pair $(\mathbf{M}, \omega)$ where ω is a state on $\mathbf{M}$. To this end let us define the following map

$$\mathbf{M} \ni X \mapsto R^{\frac{1}{2p}} X R^{\frac{1}{2p}} \tag{16}$$

where R is the operator determined by the equality $\omega(X) = \varphi(RX)$ (see also the argument leading to formula (12). One can show (cf [20]) that (16) can be extended to an isometric isomorphism τ_p between $L^p(\mathbf{M}, \omega)$ and $L^p(\mathbf{M}, \varphi)$. Moreover, let V be an isometry from $L^p(\mathbf{M}, \omega)$ onto $L^p(\mathbf{M}, \omega)$. Then, there exists an isometry T on $L^p(\mathbf{M}, \varphi)$ such that the following diagram is commutative

$$\begin{array}{ccc} L^p(\mathbf{M}, \varphi) & \xrightarrow{T} & L^p(\mathbf{M}, \varphi) \\ \uparrow \tau_p & & \uparrow \tau_p \\ L^p(\mathbf{M}, \omega) & \xrightarrow{V} & L^p(\mathbf{M}, \omega) \end{array}$$

Consequently, any isometry from $L^p(\mathbf{M}, \omega)$ onto $L^p(\mathbf{M}, \omega)$ has the following form:

$$V(X) = R^{-\frac{1}{2p}} T(R^{\frac{1}{2p}} X R^{\frac{1}{2p}}) R^{-\frac{1}{2p}} \tag{17}$$

Now, let us restrict ourselves to the setting of Dirac's Quantum Mechanics. So, we put $\mathbf{M} \equiv \mathcal{B}(\mathcal{H})$ and consider a linear map $V : L^p(\mathcal{B}(\mathcal{H}), \omega) \to L^p(\mathcal{B}(\mathcal{H}), \omega)$ where $\omega(\cdot) = Tr\{\varrho \cdot\}$. We assume that V is an isometry of $L^p(\mathcal{B}(\mathcal{H}), \omega)$ onto itself such that $V(\mathbf{1}) = \mathbf{1}$ and $V(X) \geq 0$, for $X \geq 0$. Then

$$V(X) = \varrho^{-\frac{1}{2p}} W B J(\varrho^{\frac{1}{2p}} X \varrho^{\frac{1}{2p}}) \varrho^{-\frac{1}{2p}} = \varrho^{-\frac{1}{2p}} J(\varrho^{\frac{1}{2p}} X \varrho^{\frac{1}{2p}}) \varrho^{-\frac{1}{2p}} = J(X)\,, \tag{18}$$

for $X \in \mathcal{B}(\mathcal{H}) \cap L^p(\mathcal{B}(\mathcal{H}), \omega)$, where the second equality follows from the positivity and identity preserving assumption as well as from the irreducibility of $\mathcal{B}(\mathcal{H})$ while the third equality follows from the fact that each Jordan isomorphism can be split into the sum of *-isomorphism and *-anti-isomorphism.

Having such a version of the converse of quantum Banach-Lamperti theorem we are in a position to discuss the questions related to *a change of representation* (cf. Section 2) but, now, in a quantum mechanical setting. Let the dynamical map $V_t : L^2(\mathcal{B}(\mathcal{H}), \omega) \to L^2(\mathcal{B}(\mathcal{H}), \omega)$ be induced by a hamiltonian flow. Can we change the representation, that is to find a map $\Lambda : L^2(\mathcal{B}(\mathcal{H}), \omega) \to L^2(\mathcal{B}(\mathcal{H}), \omega)$ such that there is no information lost in the sense that Λ (so also Λ^{-1}) is affine isomorphism of the set of states $\mathcal{S}$ onto itself and with the property that the composition : $\Lambda \circ V_t \circ \Lambda^{-1}$ no longer induced by a point dynamics? To answer this question let us note that the predual space $\mathcal{A}_*$ is isometric to $L^1(\mathcal{B}(\mathcal{H}), \omega)$ (so we can identify them), $L^1(\mathcal{B}(\mathcal{H}), \omega) \subset L^2(\mathcal{B}(\mathcal{H}), \omega)$, $\mathcal{S} \hookrightarrow L^1(\mathcal{B}(\mathcal{H}), \omega)$, and $\mathcal{S}$ is not subset of any hyperplane of $L^1(\mathcal{B}(\mathcal{H}), \omega)$. Then using Kadison's result (see [19]) one can see firstly that the affine isomorphism Λ_0 can be extended to a linear map Λ on $L^1(\mathcal{B}(\mathcal{H}), \omega)$. Secondly Λ as a map of $\mathcal{B}(\mathcal{H})_*$ onto itself is induced by a Jordan automorphism $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. Finally using the quantum converse to Koopman's theorem we infer that the composition $\Lambda \circ V_t \circ \Lambda^{-1}$ is induced by the uniquely determined Jordan automorphism. In other words, we obtained the negative answer to the above posed question.

The important point to note here is that we assumed Λ to be an affine isomorphism while V_t to be an isometry *onto*. To get the implementability of $\Lambda \circ V_t \circ \Lambda^{-1}$ by a uniquely determined Jordan morphism $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ these conditions can not be weaken. In other words, relaxing the conditions *"isomorphism"* and *"onto"* the implementability will be lost and again, we have analogous situation to that in classical case. The question of the change of representation through a non-isomorphic operator Λ and the related question of implementability will be discussed in a forthcoming publication.

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